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# Existence of solutions of a nonlinear wave equation as limits of minimisers to elliptic energies

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# 1 Introduction

The paper of Enrico Serra and Paolo Tilli [2] proves a conjecture by Ennio De Giorgi, which states that certain weak solutions of nonlinear wave equations of the form $w'' + \Delta w + |w|^{p-2}w = 0$ can be obtained as limits of functions that minimise suitable functionals. The conjecture was published in 1996 in the *Duke Mathematical Journal* [1].

**Conjecture 1.1.** (De Giorgi.) *Let $k > 1$ be an integer number. For $\varepsilon > 0$, let $v_\varepsilon(t, x)$ denote the minimiser of the convex functional*

$$F_\varepsilon(v) := \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}\Big\{|v''(t,x)|^2 + \frac{1}{\varepsilon^2}|\nabla v(t,x)|^2 + \frac{1}{\varepsilon^2}|v(t,x)|^{2k}\Big\}dxdt,$$

*subject to the boundary conditions*

$$v(0,x) = \alpha(x), \quad v'(0,x) = \beta(x), \quad x \in \mathbb{R}^n, \tag{1}$$

*where $\alpha, \beta \in C_0^\infty(\mathbb{R}^n)$ are given functions. Then, for almost every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$, the limit*

$$w(t,x) = \lim_{\varepsilon\to 0} v_\varepsilon(t,x),$$

*exists and the function $w(t, x)$ solves in $\mathbb{R}^+ \times \mathbb{R}^n$ the nonlinear wave equation*

$$w'' = \Delta w - kw^{2k-1}, \tag{2}$$

*with initial conditions*

$$w(0,x) = \alpha(x), \quad w'(0,x) = \beta(x), \quad x \in \mathbb{R}^n. \tag{3}$$

This conjecture shows that we can approximate some hyperbolic problems with suitable elliptic problems. It is also important to notice that this conjecture gives the existence for the solution of the nonlinear wave equation for any $k > 1$, and even for supercritical $k$, where uniqueness is not known.

The report is organised as follows. In Section 1, we will state the main theorem of the paper. In Section 2, we introduce the time scaling and the spaces that we will use then in Section 3 where we prove the main theorem. Finally in Section 4 we will prove some technical propositions and lemmata.

## 1.1 Notation

Let us explain the notation we will use in this report. In what follows,

- Let $a \in \mathbb{R}^n$, $r \in \mathbb{R}_+$, then $B_r(a)$ denotes the open unit ball centered in $a$.
- For $w(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$, let $w'(t,x) := \partial_t w$ be the first derivative in $t$ of $w$, and then $w'', w''', \ldots$ the higher derivatives in $t$ of $w$.
- $\Omega \subset \mathbb{R}^n$ will denote an open set.
- For $m \in \mathbb{N}, m > 0$, let

$$C^0(\Omega) := \Big\{f : \Omega \to \mathbb{R} : \ f \text{ is continuous on } \Omega\Big\},$$
$$C^m(\Omega) := \Big\{f \in C^0(\Omega) : \forall \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leq m, D^\alpha f \text{ is continuous on } \Omega\Big\},$$
$$C^\infty(\Omega) := \Big\{f : \Omega \to \mathbb{R} : f \in C^m(\Omega), \forall m \in \mathbb{N}\Big\}.$$

  For $f \in C^0(\Omega)$, we can define the norm

$$\|f\|_\infty := \sup_{x\in\Omega}\Big\{|f(x)|\Big\}.$$

- $C_0^\infty(\Omega) = \mathcal{D}$ denote the set of compactly supported and infinitely differentiable functions and $\mathcal{D}'$ its dual.

- Let

$$\text{ess sup } f = \inf\Big\{a \in \mathbb{R} : f(x) < a \text{ for a.e. } x \in \Omega\Big\}.$$

- For $1 \leq p \leq \infty$, define

$$\|f\|_{L^p} = \begin{cases} \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \text{ess sup } |f|, & \text{if } p = \infty. \end{cases}$$

  Let then

$$L^p(\Omega) := \Big\{f : \|f\|_{L^p(\Omega)} < +\infty\Big\}.$$

- For $m \in \mathbb{N}$, $1 \leq p \leq \infty$, let

$$W^{m,p}(\Omega) := \Big\{f \in L^p(\Omega) : \forall \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leq m, \text{ the weak derivative } D^\alpha f \in L^p(\Omega)\Big\},$$

$$H^m(\Omega) := \Big\{f \in L^2 : \forall \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leq m, D^\alpha f \in L^2(\Omega)\Big\}.$$

- For $1 \leq p \leq \infty$, let

$$L^p_{loc}(\Omega) = \Big\{f : \Omega \to \mathbb{R} : \forall K \subset \Omega \text{ compact}, f \in L^p(K)\Big\}.$$

## 1.2 Main theorem

Let us analyse how the minimisers $v_\varepsilon$ of $F_\varepsilon$ are related to the non linear wave equation (2) and what kind of compactness they need to converge to a solution of the nonlinear wave equation. For every $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, we can show that $v_\varepsilon$, the minimiser of $F_\varepsilon$, satisfies the Euler-Lagrange equation

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}(\varepsilon^2 v_\varepsilon'' \eta'' + \nabla v_\varepsilon \nabla \eta + \frac{p}{2}|v_\varepsilon|^{p-2} v_\varepsilon \eta) dxdt = 0.$$

Then, given an arbitrary $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, we have, letting $\eta(t,x) = e^{t/\varepsilon}\varphi(t,x)$,

$$-\int_0^\infty \int_{\mathbb{R}^n} v_\varepsilon'(\varepsilon^2\varphi'' + 2\varepsilon\varphi' + \varphi)' dxdt + \int_0^\infty \int_{\mathbb{R}^n} (\nabla v_\varepsilon \nabla \varphi + \frac{p}{2}|v_\varepsilon|^{p-2} v_\varepsilon \varphi) dxdt = 0.$$

We need that $v_\varepsilon$ converges when $\varepsilon \to 0$ to a function $w$ that satisfies the non linear wave equation (2) (with $p = 2k$), i.e. if we derive its Euler-Lagrange equation :

$$\int_0^\infty \int_{\mathbb{R}^n} (w'\varphi' + \nabla w \nabla \varphi + \frac{p}{2}|w|^{p-2} w \varphi) dxdt = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n). \tag{4}$$

Let us analyse which kind of compactness is required on $v_\varepsilon$ in order for any limit point $w$ of $\{v_\varepsilon\}_{\varepsilon>0}$ with $\varepsilon \to 0$ to satisfy (4). The first two terms need only a weak convergence (up to some subsequence) on $v_\varepsilon$ and $\nabla v_\varepsilon$. Indeed, if passing to a subsequence $\{v_{\varepsilon_i}\}_{i\in\mathbb{N}}$, with $\varepsilon_i \to 0$ when $i \to \infty$

$$v_{\varepsilon_i} \rightharpoonup w \quad \text{in} \quad H^1((0,T) \times \mathbb{R}^n),$$

then

$$-\int_0^\infty \int_{\mathbb{R}^n} v_{\varepsilon_i}'(\varepsilon_i^2\varphi'' + 2\varepsilon_i\varphi' + \varphi)' dxdt + \int_0^\infty \int_{\mathbb{R}^n} \nabla v_{\varepsilon_i} \nabla \varphi dxdt \to \int_0^\infty \int_{\mathbb{R}^n} (-w'\varphi' + \nabla w \nabla \varphi) dxdt.$$

The third term is not linear so we need $v_\varepsilon$ to converge locally strongly in $L^{p-1}$ to $w$ (up to some subsequence). In particular, if for all $T$ and $j$, passing to a subsequence $v_{\varepsilon_i}$,

$$v_{\varepsilon_i} \to w \quad \text{in} \quad L^{p-1}((0,T) \times B_j(0)),$$

then

$$-\int_0^\infty \int_{\mathbb{R}^n} |v_\varepsilon|^{p-2} v_\varepsilon \varphi dxdt \to \int_0^\infty \int_{\mathbb{R}^n} |w|^{p-2} w \varphi dxdt.$$

Thus we get

$$\int_0^\infty \int_{\mathbb{R}^n} w''\varphi + \nabla w \nabla \varphi + \frac{p}{2}|w|^{p-2} w \varphi dxdt = 0,$$

and $w$ satisfies the Euler-Lagrange equation (4).

This leads us to the main theorem of the paper of Serra and Tilli [2].

**Theorem 1.2.** *For $p \geq 2$ and $\varepsilon > 0$, let $v_\varepsilon(t,x)$ denote the unique minimiser of the strictly convex functional*

$$F_\varepsilon(v) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}\Big\{|v''(t,x)|^2 + \frac{1}{\varepsilon^2}|\nabla v(t,x)|^2 + \frac{1}{\varepsilon^2}|v(t,x)|^p\Big\}dxdt,$$

*under the boundary conditions* (1)*, where $\alpha$ and $\beta$ are given functions such that*

$$\alpha, \beta \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

*Then*

1. Estimates. *There exists a constant $C > 0$ (which depends only on $\alpha,\beta$, $p$ and $n$) such that, for every $\varepsilon \in (0,1)$,*
$$\int_0^T \int_{\mathbb{R}^n} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^p)dxdt \leq CT, \quad \forall T \geq \varepsilon, \tag{5}$$
$$\int_{\mathbb{R}^n} |v_\varepsilon'(t,x)|^2 dx \leq C, \quad \int_{\mathbb{R}^n} |v_\varepsilon(t,x)|^2 dx \leq C(1+t^2), \quad \forall t \geq 0, \tag{6}$$
*and, for every function $h \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,*
$$\left|\int_{\mathbb{R}^n} v_\varepsilon''(t,x)h(x)dx\right| \leq C(\|h\|_{L^p(\mathbb{R}^n)} + \|\nabla h\|_{L^2(\mathbb{R}^n)}), \quad \text{for a.e. } t > 0. \tag{7}$$

2. Convergence. *Every sequence $v_{\varepsilon_i}$ (with $\varepsilon_i \to 0$) admits a subsequence that is convergent, in the strong topology of $L^q((0,T)\times A)$ for every $T > 0$ and every bounded open set $A \subset \mathbb{R}^n$ (with arbitrary $q \in [2,p)$ if $p > 2$, and with $q = 2$ if $p = 2$), almost everywhere on $\mathbb{R}^+ \times \mathbb{R}^n$, and in the weak topology of $H^1((0,T),\mathbb{R}^n)$ for every $T > 0$, to a function $w$ such that*
$$w \in L^\infty(\mathbb{R}^+; L^p(\mathbb{R}^n)), \quad \nabla w \in L^\infty(\mathbb{R}^+; L^p(\mathbb{R}^n)), \tag{8}$$
$$w' \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^n)), \quad w \in L^\infty((0,T); H^1(\mathbb{R}^n)) \quad , \forall T > 0, \tag{9}$$
*which solves in $\mathbb{R}^+ \times \mathbb{R}^n$ the nonlinear wave equation*
$$w'' = \Delta w - \frac{p}{2}|w|^{p-2}w. \tag{10}$$
*with initial conditions as in* (3).

3. Energy inequality. *Letting*
$$\mathcal{E}(t) = \int_{\mathbb{R}^n} (|w'(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^p)dx, \tag{11}$$
*the function w(t,x) satisfies the energy inequality*
$$\mathcal{E}(t) \leq \mathcal{E}(0) = \int_{\mathbb{R}^n} (|\beta(x)|^2 + |\nabla\alpha(x)|^2 + |\alpha(x)|^p)dx, \quad \text{for a.e. } t > 0. \tag{12}$$

## 2 Time scaling

In
$$F_\varepsilon(v) = \int_0^\infty \int_{\mathbb{R}^n} e^{-s/\varepsilon}\Big\{|v''(s,x)|^2 + \frac{1}{\varepsilon^2}|\nabla v(s,x)|^2 + \frac{1}{\varepsilon^2}|v(s,x)|^p\Big\}dxds,$$
the dependence in $\varepsilon$ in the exponential will be an issue for the estimations. Therefore, the idea is to rescale the integral letting $t = s/\varepsilon$ and $u(t,x) = v(\varepsilon t, x)$ in order to get rid of the dependence in $\varepsilon$ in the exponential.
For $\varepsilon > 0$, we let
$$J_\varepsilon(u) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t}\{|u''(t,x)|^2 + \varepsilon^2|\nabla u(t,x)|^2 + \varepsilon^2|u(t,x)|^p\}dxdt.$$
We have $F_\varepsilon(v) = \varepsilon^{-3}J_\varepsilon(u)$. The minimisation of $F_\varepsilon(v)$ under the initial conditions (3) is equivalent to the minimisation of $J_\varepsilon(u)$ under the initial conditions
$$u(0,x) = \alpha(x), \quad u'(0,x) = \varepsilon\beta(x).$$

Let $\mathcal{U}$ be the set of (equivalence classes of) functions $u \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^n)$ such that

$$\nabla u \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^n), \quad u'' \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^n),$$

and such that

$$\|u\|_{\mathcal{U}} := \left(\int_0^\infty \int_{\mathbb{R}^n} e^{-t}\{|u''(t,x)|^2 + |\nabla u(t,x)|^2\}dxdt\right)^{1/2} + \left(\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|u(t,x)|^p dxdt\right)^{1/p} < +\infty.$$

Note that $\mathcal{U}$, endowed with the norm $\|\cdot\|_{\mathcal{U}}$, is a Banach space. Let

$$\mathcal{U}^\varepsilon_{\alpha,\beta} = \{u \in \mathcal{U} : u(0,x) = \alpha(x), \quad u'(0,x) = \varepsilon\beta(x)\}.$$

Notice that $\mathcal{U}^\varepsilon_{\alpha,\beta}$ is a closed and convex subset of $\mathcal{U}$. As $\mathcal{U}^\varepsilon_{\alpha,\beta} \neq \emptyset$ (consider for instance $u(t,x) = \alpha(x) + \varepsilon t\beta(x)$), we can show that $J_\varepsilon(u)$ admits a unique minimiser in $\mathcal{U}^\varepsilon_{\alpha,\beta}$. We will always denote by $u_\varepsilon(t,x)$ the minimiser of $J_\varepsilon(u)$ in $\mathcal{U}^\varepsilon_{\alpha,\beta}$.

# 3 Proof of the main Theorem

In this section, we will sketch the proof of Theorem 1.2. Therefore we will need some definitions and results that we will show in Section 4. Define

$$L_\varepsilon(t) := \int_{\mathbb{R}^n}\left(|u''_\varepsilon(t,x)|^2 + \varepsilon^2|\nabla u_\varepsilon(t,x)|^2 + \varepsilon^2|u_\varepsilon(t,x)|^p\right)dx, \quad \text{for a.e. } t > 0. \tag{13}$$

$$H_\varepsilon(t) := \int_t^\infty e^{-s}L_\varepsilon(s)ds, \quad t \geq 0. \tag{14}$$

Note that $H_\varepsilon$ is continuous, nonnegative and nonincreasing, and $H_\varepsilon(0) = J_\varepsilon(u)$. In particular, we see that $e^{-t}L_\varepsilon(t) \in L^1(\mathbb{R}^+)$ and $H \in W^{1,1}((0,T))$ for every $T > 0$, with

$$H'_\varepsilon(t) = -e^{-t}L_\varepsilon(t), \quad \text{for a.e. } t > 0. \tag{15}$$

Further, let

$$D_\varepsilon(t) := \int_{\mathbb{R}^n}|u''_\varepsilon(t,x)|^2dx, \quad \text{for a.e. } t > 0, \tag{16}$$

and

$$I_\varepsilon(t) := \frac{1}{2}\int_{\mathbb{R}^n}|u'_\varepsilon(t,x)|^2dx, \quad \forall t \geq 0. \tag{17}$$

By Lemma 3.1 below, $e^{-t}I_\varepsilon(t)$ is in $W^{1,1}(\mathbb{R}^+)$, so $\lim_{T\to\infty} e^{-T}I_\varepsilon(T) = 0$, $I_\varepsilon$ is in $W^{1,1}(0,T)$ for all $T > 0$ and

$$I'_\varepsilon(t) = \int_{\mathbb{R}^n} u'_\varepsilon(t,x)u''_\varepsilon(t,x)dx, \quad \text{for a.e. } t > 0. \tag{18}$$

Moreover

$$(e^{-t}I_\varepsilon(t))' = -e^{-t}I_\varepsilon(t) + e^{-t}I'_\varepsilon(t), \quad \text{for a.e. } t. \tag{19}$$

Finally, set

$$\begin{aligned} E_\varepsilon(t) &:= I_\varepsilon(t) - I'_\varepsilon(t) + \frac{e^t}{2}H_\varepsilon(t) \\ &= \frac{1}{2}\int_{\mathbb{R}^n}|u'_\varepsilon(t,x)|^2dx - \int_{\mathbb{R}^n}u'_\varepsilon(t,x)u''_\varepsilon(t,x)dx + \frac{e^t}{2}\int_t^\infty e^{-s}L_\varepsilon(s)ds, \quad \text{for a.e. } t \geq 0. \end{aligned} \tag{20}$$

We assume for the moment the following result, that we will prove in Section 4.

**Lemma 3.1.** *There exists a constant $C = C(\alpha,\beta)$ such that the minimiser $u_\varepsilon$ satisfies*

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|u''_\varepsilon|^2dxdt \leq C\varepsilon^2, \tag{21}$$

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|u'_\varepsilon|^2dxdt \leq C\varepsilon^2, \tag{22}$$

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|u_\varepsilon|^2dxdt \leq C, \tag{23}$$

$$\int_0^\infty \int_{\mathbb{R}^+} e^{-t}|u'_\varepsilon(t,x)||u''_\varepsilon(t,x)|dxdt \leq C\varepsilon^2. \tag{24}$$

## 3.1 The function $E_\varepsilon$

In this section, we will explain why the function $E_\varepsilon$ is central for the proof. First let state some interesting results for $E_\varepsilon$.

**Proposition 3.2.** *The function $E_\varepsilon(t)$ satisfies*

$$E'_\varepsilon = -2D_\varepsilon \text{ in } \mathcal{D}'(\mathbb{R}^+).$$

*In particular, the function $E_\varepsilon(t)$ is nonincreasing, belongs to $W^{1,1}((0,T))$ for every $T > 0$, and therefore it admits a continuous representative.*

The proof of this Proposition is quite technical and will be given in Section 4.

**Lemma 3.3.** *The function $E_\varepsilon(t)$ is nonnegative. More precisely,*

$$I_\varepsilon(t) + \frac{e^t}{2}\int_t^\infty H_\varepsilon(s)ds \le E_\varepsilon(t), \quad \forall t \ge 0. \tag{25}$$

*Proof.* From (19), for almost every $s > 0$ we have that

$$-(e^{-t}I_\varepsilon(s))' + \frac{1}{2}H_\varepsilon(s) = e^{-s}E_\varepsilon(s),$$

and hence, for any two numbers $T > t \ge 0$, integration over $(t, T)$ yields

$$\begin{aligned} e^{-t}I_\varepsilon(t) - e^{-T}I_\varepsilon(T) + \frac{1}{2}\int_t^T H_\varepsilon(s)ds &= \int_t^T e^{-s}E_\varepsilon(s)ds \\ &\le E_\varepsilon(t)\int_t^T e^{-s}ds = E_\varepsilon(t)\Big(e^{-t} - e^{-T}\Big), \end{aligned}$$

since $E_\varepsilon$ is nonincreasing (see Proposition 3.2). Letting $T \to \infty$ and using that $\lim_{T\to\infty} e^{-T}I_\varepsilon(T) = 0$, we get the result.

Finally, the following lemma (that we will not prove in this report) gives a good estimate of $E_\varepsilon$.

**Lemma 3.4.** *There exists a constant $C = C(\alpha, \beta) > 0$ such that*

$$0 \le E_\varepsilon(t) \le E_\varepsilon(0) \le C\varepsilon^2, \quad \forall t \ge 0, \tag{26}$$

*where by $E_\varepsilon(0)$ we mean the value at $t = 0$ of the continuous representative of $E_\varepsilon(t)$.*

We introduced the function $E_\varepsilon$ because it bounds from above $I_\varepsilon$ and $H_\varepsilon$ from Lemma 3.3 and it has good properties. In particular $E_\varepsilon$ is nonincreasing by Proposition 3.2 and can be well estimated by Lemma 3.4. From Lemma 3.3 and 3.4, we obtain the following:

**Theorem 3.5.** *There exists a constant $C = C(\alpha, \beta) > 0$ such that*

$$\int_t^{t+1}\int_{\mathbb{R}^n}(|\nabla u_\varepsilon(s, x)|^2 + |u_\varepsilon(s, x)|^p)dxds \le C, \quad \forall t \ge 0, \tag{27}$$

*and*

$$2I_\varepsilon(t) = \int_{\mathbb{R}^n}|u'_\varepsilon(t, x)|^2dx \le C\varepsilon^2, \quad \forall t \ge 0. \tag{28}$$

Note that (25) breaks in two pieces:

$$\frac{e^t}{2}\int_t^\infty H_\varepsilon(s)ds \le E_\varepsilon(t), \quad \forall t \ge 0, \tag{29}$$

and

$$I_\varepsilon(t) \le E_\varepsilon(t), \quad \forall t \ge 0. \tag{30}$$

Then, (27) follows from (29) and (26), while (28) follows directly from (30) and (26). Moreover, roughly speaking, by (27), two of the three terms of $J_\varepsilon(u_\varepsilon)$ are bounded, so we can show that the last term is bounded :

**Theorem 3.6.** *There exists $C = C(\alpha, \beta) > 0$ such that, for every function $h$ in $H^1 \cap L^p$,*

$$\left|\int_{\mathbb{R}^n} u''_\varepsilon(t, x)h(x)dx\right| \le C\varepsilon^2(\|h\|_{L^p} + \|\nabla h\|_{L^2}), \quad \text{for a.e. } t > 0,$$

### 3.2 Estimates

Here we prove the estimations on the minimisers $v_\varepsilon(t,x)$ of $F_\varepsilon$ mentioned in Theorem 1.2, i.e. (5), (6) and (7). Note that

$$u_\varepsilon(t,x) = v_\varepsilon(\varepsilon t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{31}$$

and

$$v_\varepsilon(0,x) = \alpha(x), \quad v'_\varepsilon(0,x) = \beta(x). \tag{32}$$

We get (7) directly from Theorem 3.6, using (31). In order to prove (5), let us rewrite (27) in terms of $v_\varepsilon$ using (31) :

$$\int_t^{t+1} \int_{\mathbb{R}^n} (|\nabla v_\varepsilon(\varepsilon s, x)|^2 + |v_\varepsilon(\varepsilon s, x)|^p) dx ds \leq C, \quad \forall t \geq 0.$$

now, substituting $\sigma = \varepsilon s$,

$$\int_{\varepsilon t}^{\varepsilon t+\varepsilon} \int_{\mathbb{R}^n} \frac{1}{\varepsilon} (|\nabla v_\varepsilon(\sigma, x)|^2 + |v_\varepsilon(\sigma, x)|^p) dx d\sigma \leq C, \quad \forall t \geq 0.$$

Since $t$ is arbitrary, we can rename $\tau = \varepsilon t$ :

$$\int_{\tau}^{\tau+\varepsilon} \int_{\mathbb{R}^n} (|\nabla v_\varepsilon(\sigma, x)|^2 + |v_\varepsilon(\sigma, x)|^p) dx d\sigma \leq C\varepsilon, \quad \forall \tau \geq 0.$$

Now we want to integrate over $[0,T]$ with $T > \varepsilon$. So let us take a set of $\lceil T/\varepsilon \rceil$ intervals of the form $[t, t+\varepsilon]$ that covers all $[0,T]$. Then we can just sum the integral over those intervals to get an estimation for the integral over $[0,T]$:

$$\int_0^T \int_{\mathbb{R}^n} (|\nabla v_\varepsilon(\sigma, x)|^2 + |v_\varepsilon(\sigma, x)|^p) dx d\sigma \leq C\varepsilon \lceil T/\varepsilon \rceil \leq 2CT, \quad \forall T > \varepsilon. \tag{33}$$

This proves (5). In order to show (6), let us use (28) with $t/\varepsilon$ instead of $t$:

$$\int_{\mathbb{R}^n} |u'_\varepsilon(t/\varepsilon, x)|^2 dx = \int_{\mathbb{R}^n} |\varepsilon v'_\varepsilon(t,x)|^2 dx \leq C\varepsilon^2,$$

which implies

$$\int_{\mathbb{R}^n} |v'_\varepsilon(t,x)|^2 dx \leq C. \tag{34}$$

This proves the first part of (6). Then,

$$v_\varepsilon(t,x) = v_\varepsilon(0,x) + \int_0^t v_\varepsilon(s,x) ds,$$

and as $(a+b)^2 \leq 2(a^2+b^2) \quad \forall (a,b) \in \mathbb{R}^2$,

$$|v_\varepsilon(t,x)|^2 \leq \left( |v_\varepsilon(0,x)| + \int_0^t |v'_\varepsilon(s,x)| ds \right)^2 \leq 2|v_\varepsilon(0,x)|^2 + 2\left( \int_0^t |v'_\varepsilon(s,x)| ds \right)^2.$$

By Hölder inequality

$$|v_\varepsilon(t,x)|^2 \leq 2|v_\varepsilon(0,x)|^2 + 2t \int_0^t |v'_\varepsilon(s,x)|^2 ds.$$

By the initial conditions (3) we know that $v_\varepsilon(0,x) = \alpha(x) \in L^2(\mathbb{R}^n)$. Thus, using (34)

$$\int_{\mathbb{R}^n} |v_\varepsilon(t,x)|^2 dx \leq 2 \int_{\mathbb{R}^n} |\alpha(x)|^2 + 2t \int_0^t \int_{\mathbb{R}^n} |v'_\varepsilon(s,x)|^2 dx ds \leq C(1+t^2),$$

and (6) is proved.

### 3.3 Passage to the limit

In this Section, we will show the Convergence part of Theorem 1.2, i.e. (8), (9) and (10).

### 3.3.1 Euler-Lagrange equation

We will show here that we have the compactness conditions required in the discussion from Section 1.2. Let $Q^T$ denote the cylinder $(0,T)\times\mathbb{R}^n$. From the first part of the theorem ((5) and (6)), for all $T>0$ there exists $C_T>0$ such that

$$\|v_\varepsilon\|_{H^1(Q^T)}+\|v_\varepsilon\|_{L^p(Q^T)}\le C_T.$$

Take the family of balls $\{B_j(0)\}_{j\in\mathbb{N}}$, and let $A_j^T=(0,T)\times B_j(0)$. Note that $v_\varepsilon\in H^1(Q^T)$ implies $v_\varepsilon\in L^2(A_j^T)$ for all $j$ and $T>0$. For $p>2$, we also have that the $v_\varepsilon$ are equibounded in $L^p(A_j^T)$ for every $T>0$ and $j$. As a consequence, by a diagonal argument and up to passing to a subsequence, we can assume that for every $T>0$ and every $j$,

$$v_\varepsilon\rightharpoonup w \quad \text{in} \quad H^1(Q^T), \tag{35}$$

$$v_\varepsilon\rightharpoonup w \quad \text{in} \quad L^p(Q^T), \tag{36}$$

$$v_\varepsilon\to w \quad \text{in} \quad L^q(A_j^T) \quad \text{for every } q\in[2,p),\ (\text{for } q=2,\ \text{if } p=2), \tag{37}$$

for a suitable function $w$ that belongs to $H^1(Q^T)\cap L^p(Q^T)$ for all $T>0$. By the discussion from Section 1.2, we get that $w$ is a weak solution, in $\mathbb{R}^+\times\mathbb{R}^n$, of the nonlinear wave equation (10), as claimed in Theorem 1.2.

### 3.3.2 $L^\infty$ bounds

Here we will prove (8) and (9). Letting $\varepsilon\to0$ in (33), by lower semicontinuity we get

$$\int_t^{t+T}\int_{\mathbb{R}^n}\Big(|\nabla w|^2+|w|^p\Big)\le CT,\quad \forall t\ge0.$$

Dividing by $T$ and letting $T\to0$, one obtains (8). Moreover, letting $\varepsilon\to0$ in (6), by lower semicontinuity we have

$$\int_{\mathbb{R}^n}|w'(t,x)|^2dx\le C,\qquad \int_{\mathbb{R}^n}|w(t,x)|^2dx\le C(1+t^2),\quad \text{for a.e. } t>0,$$

which yields (9).

### 3.3.3 Initial conditions

Now we want to show that $w$ satisfies the initial conditions (3). Let us prove the first condition. For all $\varepsilon$, for all $a(t)\in C_0^\infty(\mathbb{R})$ and $h(x)\in L^2(\mathbb{R}^n)$, and using (6) and the first condition of (32),

$$\begin{aligned}-a(0)\int_{\mathbb{R}^n}\alpha(x)h(x)dx &= -\int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n}v_\varepsilon(t,x)h(x)dx\Big)dt-a(0)\int_{\mathbb{R}^n}\alpha(x)h(x)dx\\ &\quad+\int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n}v_\varepsilon(t,x)h(x)dx\Big)dt\\ &=\int_0^\infty a(t)\Big(\int_{\mathbb{R}^n}v_\varepsilon'(t,x)h(x)dx\Big)dt+\int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n}v_\varepsilon(t,x)h(x)dx\Big)dt,\end{aligned}$$

by integration by parts. Now letting $\varepsilon\to0$, and using (35) we get

$$\begin{aligned}-a(0)\int_{\mathbb{R}^n}\alpha(x)h(x)dx &=\int_0^\infty a(t)\Big(\int_{\mathbb{R}^n}w'(t,x)h(x)dx\Big)dt+\int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n}w(t,x)h(x)dx\Big)dt\\ &=\int_0^\infty\frac{d}{dt}\Big(a(t)\int_{\mathbb{R}^n}w(t,x)h(x)dx\Big)dt\\ &=-a(0)\int_{\mathbb{R}^n}w(0,x)h(x)dx,\end{aligned}$$

so from the arbitrariness of $a\in C_0^\infty(\mathbb{R})$ and $h\in L^2(\mathbb{R}^n)$, we have that $w(0,x)=\alpha(x)$. For the second condition, let $X$ denote the Banach space $H^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$. Given $\varphi\in L^2(\mathbb{R}^n)$, we can regard $\varphi$ as an element of $X'$ letting

$$\langle\varphi,h\rangle=\int_{\mathbb{R}^n}\varphi(x)h(x)dx,$$

via the natural embedding $L^2\hookrightarrow X'$. In this sense, the first part of (6) provides a bound for $v_\varepsilon'$ in $L^\infty(\mathbb{R}^+;X')$ and (7) gives a bound for $v_\varepsilon''$ in $L^\infty(\mathbb{R}^+;X')$. Passing to a subsequence, we may then assume that $v_\varepsilon'\to\psi_1$ and

$v''_\varepsilon \to \psi_2$ weakly-* in $L^\infty(\mathbb{R}^+; X')$ for suitable $\psi_1, \psi_2 \in L^\infty(\mathbb{R}^+; X')$. By (37) and the uniqueness of the limit, $\psi_1 = w'$ and $\psi_2 = w''$. In particular, $w' \in W^{1,\infty}(\mathbb{R}^+; X')$ and $w'(0)$ is well defined as an element of $X'$. Now, given $h \in X$ and $a \in C_0^\infty(\mathbb{R})$, from the second condition of (33) we have, for all $\varepsilon$,

$$\begin{aligned}-a(0)\langle\beta(x), h(x)\rangle &= -\int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n} v'_\varepsilon(t,x)h(x)dx\Big)dt - a(0)\int_{\mathbb{R}^n}\beta(x)h(x)dx + \int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n} v'_\varepsilon(t,x)h(x)dx\Big)dt\\ &= \int_0^\infty a(t)\Big(\int_{\mathbb{R}^n} v''_\varepsilon(t,x)h(x)dx\Big)dt + \int_0^\infty a'(t)\Big(\int_{\mathbb{R}^n} v'_\varepsilon(t,x)h(x)dx\Big)dt.\end{aligned}$$

Letting $\varepsilon \to 0$, since $a(t)h(x)$ belongs to $L^1(\mathbb{R}^+; X)$, we obtain that

$$\begin{aligned}-a(0)\int_{\mathbb{R}^n}\beta(x)h(x) &= \int_{\mathbb{R}^n} a(t)\Big(\int_0^\infty w''(t,x)h(x)dx\Big)dt + \int_{\mathbb{R}^n} a'(t)\Big(\int_0^\infty w'(t,x)h(x)dx\Big)dt\\ &= \int_0^\infty \frac{d}{dt}\Big(a(t)\int_{\mathbb{R}^n} w'(t,x)h(x)dx\Big)dt\\ &= -a(0)\int_{\mathbb{R}^n} w'(0,x)h(x)dx,\end{aligned}$$

which, from the arbitrariness of $a \in C_0^\infty(\mathbb{R})$ and $h \in X$, proves that $w'(0) = \beta$, this inequality being meant in $X'$. This yields (3).

## 3.4 Energy inequality

Let us prove the energy inequality (12). We will need a technical lemma and a proposition that we will not prove

**Lemma 3.7.** *Let $l(t)$, $m(t)$ be nonnegative functions in $L^1_{loc}$ such that*

$$e^t\int_t^\infty\int_s^\infty e^{-z}l(z)dzds \le m(t), \quad \textit{for a.e. } t > 0.$$

*Then, for every pair of numbers $a > 0$ and $\theta \in (0,1)$,*

$$\Big(\int_0^{\theta a} se^{-s}ds\Big)\int_{T+\theta a}^{T+a} l(t)dt \le \int_T^{T+a} m(t)dt, \quad \forall T \ge 0.$$

**Proposition 3.8.** (Sharp estimate for $E(0)$). *There exists a constant $C = C(\alpha,\beta) > 0$ such that*

$$E(0) \le \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n}\Big(|\nabla\alpha(x)|^2 + |\alpha(x)|^p + |\beta(x)|^2\Big)dx + C\varepsilon^3.$$

*Proof of the energy inequality.* From the computation of $(e^{-s}I_\varepsilon(s))'$ and the definiton of $E_\varepsilon$ (see (19) and (20)),

$$-(e^{-s}I_\varepsilon(s))' + \frac{1}{2}H_\varepsilon(s) = e^{-s}E_\varepsilon(s),$$

and hence, for any two numbers $T > t \ge 0$, integration over $(t,T)$ yields

$$\begin{aligned}e^{-t}I_\varepsilon(t) - e^{-T}I_\varepsilon(T) + \frac{1}{2}\int_t^T H_\varepsilon(s)ds &= \int_t^T e^{-s}E_\varepsilon(s)ds\\ &\le E_\varepsilon(t)\int_t^T e^{-s}ds = E_\varepsilon(t)\Big(e^{-t} - e^{-T}\Big),\end{aligned}$$

since $E_\varepsilon$ is nonincreasing from Proposition 3.2 that we will discuss in Section 4. Now letting $T \to \infty$, from the fact that $e^{-t}I_\varepsilon$ is in $W^{1,1}(\mathbb{R}^+)$, one has $\lim_{T\to\infty} e^{-T}I_\varepsilon(T) = 0$, and hence

$$I_\varepsilon(t) + \frac{e^t}{2}\int_t^\infty H_\varepsilon(s)ds \le E_\varepsilon(t), \quad \forall t \ge 0.$$

This can be rearranged using $E_\varepsilon(t) \le E_\varepsilon(0)$

$$e^t\int_t^\infty H_\varepsilon(s)ds \le 2E_\varepsilon(0) - 2I_\varepsilon(t), \quad \forall t \ge 0. \tag{38}$$

Now, letting

$$l(t) = \int_{\mathbb{R}^n} \Big( |\nabla u_\varepsilon(t,x)|^2 + |u_\varepsilon|^p \Big) dx, \tag{39}$$

we have from the definitions of $L_\varepsilon$ and $H_\varepsilon$ ((13) and (14)) that

$$\frac{1}{\varepsilon^2} H_\varepsilon(s) \geq \int_s^\infty e^{-z} l(z) dz, \quad s \geq 0,$$

and hence, dividing (38) by $\varepsilon^2$, we see that we can apply Lemma 3.7 with $l(z)$ defined as above and $m(t) = 2(E_\varepsilon(0) - I_\varepsilon(t))/\varepsilon^2$. As a result of the lemma, rearranging terms we obtain the inequality

$$\Big( \int_0^{\theta a} s e^{-s} ds \Big) \int_{T+\theta a}^{T+a} l(t) dt + \frac{2}{\varepsilon^2} \int_T^{T+a} I_\varepsilon(t) dt \leq a \frac{E_\varepsilon(0)}{\varepsilon^2}, \quad \forall T > 0,$$

for arbitrary $a > 0$ and $\theta$ fixed for the moment. We can rewrite the latter inequality in the following form

$$Y(\theta a) \int_{T+\theta a}^{T+a} l(t) dt + \frac{2}{\varepsilon^2} \int_T^{T+a} I_\varepsilon(t) dt \leq a \frac{E_\varepsilon(0)}{\varepsilon^2}, \quad \forall T > 0, \tag{40}$$

where we have set for simplicity

$$Y(z) = \int_0^z s e^{-s} ds. \tag{41}$$

Now, using the defintions of $l(t)$ and $I_\varepsilon(t)$ ((39) and (17)), it is convenient to write (40) explicitly in terms of the function $u_\varepsilon$, thus obtaining

$$\begin{aligned} Y(\theta a) \int_{T+\theta a}^{T+a} \Big( \int_{\mathbb{R}^n} (|\nabla u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^p) dx \Big) dt& \\ + \frac{1}{\varepsilon^2} \int_{T+\theta a}^{T+a} \Big( \int_{\mathbb{R}^n} |u_\varepsilon'(t,x)|^2 dx \Big) dt &\leq a \frac{2E_\varepsilon(0)}{\varepsilon^2}, \quad \forall T > 0. \end{aligned}$$

Reexpressing this estimate in terms of $v_\varepsilon$ using (31) and the change of variable $s = \varepsilon t$ in the time integral we find that

$$\begin{aligned} Y(\theta a) \int_{T+\varepsilon\theta a}^{T+\varepsilon a} \Big( \int_{\mathbb{R}^n} (|\nabla v_\varepsilon(s,x)|^2 + |v_\varepsilon(s,x)|^p) dx \Big) ds& \\ + \int_{T+\varepsilon\theta a}^{T+\varepsilon a} \Big( \int_{\mathbb{R}^n} |v_\varepsilon'(s,x)|^2 dx \Big) ds &\leq a \frac{2E_\varepsilon(0)}{\varepsilon}, \quad \forall T > 0, \end{aligned}$$

where we have written $T$ instead of $\varepsilon T$ since $T$ is arbitrary. Now, for fixed $\delta > 0$, any interval $(t, t+\delta)$, with $t \geq \varepsilon\theta a$, can be covered by $\lceil \delta/((1-\theta)\varepsilon a) \rceil$ consecutive intervals of the kind $(T + \varepsilon\theta a, T + \varepsilon a)$ for suitable values of $T$; summing the corresponding inequalities as above, we find

$$\begin{aligned} Y(\theta a) \int_t^{t+\delta} \Big( \int_{\mathbb{R}^n} (|\nabla v_\varepsilon(s,x)|^2 + |v_\varepsilon(s,x)|^p) dx \Big) ds& \\ + \int_t^{t+\delta} \Big( \int_{\mathbb{R}^n} |v_\varepsilon'(s,x)|^2 dx \Big) ds &\leq a \frac{2E_\varepsilon(0)}{\varepsilon} \Big\lceil \frac{\delta}{(1-\theta)\varepsilon a} \Big\rceil, \quad \forall t \geq \varepsilon\theta a. \end{aligned}$$

Keeping $a, \theta, \delta$ and $t$ fixed, we let $\varepsilon \to 0$ in the previous inequality. Using the weak convergence of $v_\varepsilon$ in $H^1(Q^T)$ and $L^p(Q^T)$ ((35) and (36)), by lower semicontinuity we obtain that

$$\begin{aligned} Y(\theta a) \int_t^{t+\delta} \Big( \int_{\mathbb{R}^n} (|\nabla w(s,x)|^2 + |w(s,x)|^p) dx \Big) ds + \int_t^{t+\delta} \Big( \int_{\mathbb{R}^n} |w'(s,x)|^2 dx \Big) ds& \\ \leq a \limsup_{\varepsilon \to 0} \Big( \frac{2E_\varepsilon(0)}{\varepsilon} \Big\lceil \frac{\delta}{(1-\theta)\varepsilon a} \Big\rceil \Big), \quad \forall t > 0.& \end{aligned} \tag{42}$$

To estimate the limsup, observe that

$$a \lim_{\varepsilon \to 0} \Big( \varepsilon \Big\lceil \frac{\delta}{(1-\theta)\varepsilon a} \Big\rceil \Big) = \frac{\delta}{(1-\theta)a},$$

and hence, from Proposition 3.8,

$$\limsup_{\varepsilon \to 0} \Big( \frac{2E_\varepsilon(0)}{\varepsilon} \Big\lceil \frac{\delta}{(1-\theta)\varepsilon a} \Big\rceil \Big) \leq \frac{\delta}{(1-\theta)a} \int_{\mathbb{R}^n} (|\nabla \alpha(x)|^2 + |\alpha(x)|^p + |\beta(x)|^2) dx,$$

which, plugged into (42), yields

$$Y(\theta a)\int_t^{t+\delta}\Big(\int_{\mathbb{R}^n}(|\nabla w(s,x)|^2+|w(s,x)|^p)dx\Big)ds$$
$$+\int_t^{t+\delta}\Big(\int_{\mathbb{R}^n}|w'(s,x)|^2dx\Big)ds\le\frac{\delta}{1-\theta}\mathcal{E}(0),\quad\forall t>0,$$

where $\mathcal{E}(0)$ is defined according to (11). Now, dividing by $\delta$ and choosing, for instance, $\theta=\delta$ and $a=\delta^{-2}$

$$Y\Big(\frac{1}{\delta}\Big)\int_t^{t+\delta}\Big(\int_{\mathbb{R}^n}(|\nabla w(s,x)|^2+|w(s,x)|^p)dx\Big)ds$$
$$+\int_t^{t+\delta}\Big(\int_{\mathbb{R}^n}|w'(s,x)|^2dx\Big)ds\le\frac{\delta}{1-\delta}\mathcal{E}(0),\quad\forall t>0,$$

letting $\delta\to 0$ and noticing that $Y(\frac{1}{\delta})\to 1$ (see (41)), we find that, by the Lebesgue points Theorem

$$\int_{\mathbb{R}^n}(|\nabla w(t,x)|^2+|w(t,x)|^p)dx+\int_{\mathbb{R}^n}|w'(t,x)|^2dx\le\mathcal{E}(0),\quad\text{for a.e. } t>0,$$

and the validity of (11) is established.

# 4 Proofs of technical Theorems and lemmata

In this section we will prove Proposition 3.2 and Lemma 3.1
*Proof of Proposition 3.2.* Given an arbitrary $\eta\in C_0^\infty(\mathbb{R}^+)$, consider the primitive function

$$g(t)=\int_0^t\eta(s)ds. \tag{43}$$

Note that $g\in C^\infty(\mathbb{R}^+)$ and

$$g(t)\equiv 0,\quad\text{for } t \text{ close to } 0. \tag{44}$$

Let

$$\varphi(t,\delta)=t-\delta g(t).$$

We will denote by $\varphi'(t,\delta):=\partial_t\varphi(t,\delta)$ and $\varphi''(t,\delta):=\partial_{tt}\varphi(t,\delta)$. Note that $|\varphi'(t,\delta)|\ge 1-\delta\|\eta\|_\infty$ for all $t\ge 0$ so for every $\delta$ in $\mathbb{R}$ with $|\delta|$ small enough, $\varphi'(t,\delta)\ge 1/2$ and for fixed $\delta>0$ small, $\varphi$ is a diffeomorphism of $\mathbb{R}^+$ of class $C^1$. Now, for small $\delta$, we construct the competitor

$$U_\varepsilon(t,x,\delta)=u_\varepsilon(\varphi(t,\delta),x).$$

At $\delta=0$, $U_\varepsilon(t,x,0)=u_\varepsilon(t,x)$. If $U_\varepsilon\in\mathcal{U}^\varepsilon_{\alpha,\beta}$, the minimality of $u_\varepsilon$ implies

$$\frac{d}{d\delta}J_\varepsilon(U_\varepsilon(t,x,\delta))\Big|_{\delta=0}=0. \tag{45}$$

So let us first show that $U_\varepsilon\in\mathcal{U}^\varepsilon_{\alpha,\beta}$ and then derive the left hand side of (45). Using (44), we get $\varphi(t,\delta)=t$ for small $t$. Then,

$$U_\varepsilon(0,x,\delta)=\alpha(x),\quad U'_\varepsilon(0,x,\delta)=\varepsilon\beta(x).$$

Moreover,

$$U'_\varepsilon(t,x,\delta)=u'_\varepsilon(\varphi(t,\delta),x)\varphi'(t,\delta),$$
$$U''_\varepsilon(t,x,\delta)=u_\varepsilon(\varphi(t,\delta),x)|\varphi'(t,\delta)|^2+u_\varepsilon(\varphi(t,\delta),x)\varphi''(t,\delta),$$

which implies

$$J_\varepsilon(U_\varepsilon(t,x,\delta))=\int_0^\infty\int_{\mathbb{R}^n}e^{-t}\Big\{\Big(u''_\varepsilon(\varphi(t,\delta),x)|\varphi'(t,\delta)|^2+u'_\varepsilon(\varphi(t,\delta),x)\varphi''(t,\delta)\Big)^2$$
$$+\varepsilon^2|\nabla u_\varepsilon(\varphi(t,\delta),x)|^2+\varepsilon^2|u_\varepsilon(\varphi(t,\delta),x)|^p\Big\}dxdt.$$

Let $\psi = \varphi^{-1}$, the inverse being done with respect to $t$, i.e. $\psi(\varphi(t,\delta),\delta) = t$. We will denote by $\psi'(s,\delta) := \partial_s\psi(s,\delta)$. Substituting $t = \psi(s,\delta)$ in the last integral,

$$\begin{aligned} J_\varepsilon(U_\varepsilon(t,x,\delta)) = \int_0^\infty \int_{\mathbb{R}^n} \psi'(s,\delta)e^{-\psi(s,\delta)}\Big\{\Big(u_\varepsilon''(s,x)|\varphi'(\psi(s,\delta),\delta)|^2 + u_\varepsilon'(s,x)\varphi''(\psi(s,\delta),\delta)\Big)^2 \\ + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big\}dxds. \end{aligned} \tag{46}$$

Now

$$s = \varphi(\psi(s,\delta),\delta) = \psi(s,\delta) - \delta g(\psi(s,\delta)),$$

In order to show the regularity of $\psi$, we can use the implicit function theorem on the function $\Phi(t,\delta,s) := \varphi(t,\delta) - s$ noticing that

$$\partial_t\Phi(t,\delta,s) = \varphi'(t,\delta) \geq \frac{1}{2},$$

for $\delta > 0$ small enough and so $\psi$ is of class $C^\infty$. Also, we have

$$\psi(s,\delta) = s + \delta g(\psi(s,\delta)).$$

From the latter it follows that

$$\psi(s,\delta) \geq s - \delta\|g\|_\infty,$$

which implies

$$e^{-\psi(s,\delta)} \leq e^{\delta\|g\|_\infty}e^{-s}.$$

Using (21), (22), the finitness of $\|\varphi'\|_\infty$ and $\|\varphi''\|_\infty$ and the fact that $\varphi'(t)$ is compactly supported, there exists $T > 0$ such that

$$\begin{aligned} |J_\varepsilon(U_\varepsilon(t,x,\delta))| &\leq \frac{1}{1-\delta\|\eta\|_\infty}e^{\delta\|g\|_\infty}\Big\{\int_0^T\int_{\mathbb{R}^n} e^{-s}\Big(u_\varepsilon''(s,x)\|\varphi'\|_\infty^2 + u_\varepsilon'(s,x)\|\varphi''\|_\infty\Big)^2 dxds \\ &\quad + \int_0^\infty\int_{\mathbb{R}^n} e^{-s}\Big(\varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big)dxds\Big\} \\ &\leq \frac{1}{1-\delta\|\eta\|_\infty}e^{\delta\|g\|_\infty}\Big(2\|\varphi'\|_\infty^4\int_0^T\int_{\mathbb{R}^n} e^{-s}|u_\varepsilon''(s,x)|^2dxds \\ &\quad + 2\|\varphi''\|_\infty^2\int_0^T\int_{\mathbb{R}^n} e^{-s}|u_\varepsilon'(s,x)|^2dxds + J_\varepsilon(u_\varepsilon)\Big) < \infty, \end{aligned}$$

and hence $U_\varepsilon \in \mathcal{U}^\varepsilon_{\alpha,\beta}$. We want to differentiate (46) under the integral sign.

- Let us differentiate $\psi'(s,\delta)e^{-\psi(s,\delta)}$.

$$\frac{\partial}{\partial\delta}\psi(s,\delta) = g(\psi(s,\delta)) + \delta g'(\psi(s,\delta))\frac{\partial}{\partial\delta}\psi(s,\delta),$$

  and so, as $\psi(s,0) = s$ and $\psi \in C^\infty$,

$$\begin{aligned} \frac{\partial}{\partial\delta}\psi(s,\delta)\Big|_{\delta=0} &= g(s), \\ \psi'(s,\delta) &= 1 + \delta g'(\psi(s,\delta))\psi'(s,\delta), \\ \frac{\partial}{\partial\delta}\psi'(s,\delta) &= g'(\psi(s,\delta))\psi'(s,\delta) + \delta\Big(g''(\psi(s,\delta))\frac{\partial}{\partial\delta}\psi(s,\delta)\psi'(s,\delta) + g'(\psi(s,\delta))\frac{\partial}{\partial\delta}\psi'(s,\delta)\Big), \\ \frac{\partial}{\partial\delta}\psi'(s,\delta)\Big|_{\delta=0} &= g'(s). \end{aligned}$$

  Therefore

$$\frac{\partial}{\partial\delta}\Big(\psi'(s,\delta)e^{-\psi(s,\delta)}\Big)\Big|_{\delta=0} = g'(s)e^{-s} - g(s)e^{-s}.$$

- Let us now differentiate $\Big\{\Big(u_\varepsilon''(s,x)|\varphi'(\psi(s,\delta),\delta)|^2 + u_\varepsilon'(s,x)\varphi''(\psi(s,\delta),\delta)\Big)^2 + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big\}$. Since

$$\begin{aligned} \frac{\partial}{\partial\delta}|\varphi'(\psi(s,\delta))|^2 &= 2\varphi'(\psi(s,\delta),\delta)\frac{\partial}{\partial\delta}\varphi'(\psi(s,\delta),\delta), \\ \varphi'(\psi(s,\delta),\delta) &= 1 - \delta g'(\psi(s,\delta)), \\ \frac{\partial}{\partial\delta}\varphi'(\psi(s,\delta),\delta) &= -g'(\psi(s,\delta)) - \delta g''(\psi(s,\delta))\frac{\partial}{\partial\delta}\psi(s,\delta), \end{aligned}$$

it follows

$$\begin{aligned}
\frac{\partial}{\partial\delta}|\varphi'(\psi(s,\delta),\delta)|^2\Big|_{\delta=0} &= -2g'(s),\\
\varphi''(\psi(s,\delta),\delta) &= -\delta g''(\psi(s,\delta)),\\
\frac{\partial}{\partial\delta}\varphi''(\psi(s,\delta),\delta)\Big|_{\delta=0} &= -g''(s).
\end{aligned}$$

This implies

$$\begin{aligned}
&\Big\{\Big(u_\varepsilon''(s,x)|\varphi'(\psi(s,\delta),\delta)|^2 + u_\varepsilon'(s,x)\varphi''(\psi(s,\delta),\delta)\Big)^2 + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big\}\Big|_{\delta=0}\\
&= |u_\varepsilon''(s,x)|^2 + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\delta}\Big\{\Big(u_\varepsilon''(s,x)|\varphi'(\psi(s,\delta),\delta)|^2 + u_\varepsilon'(s,x)\varphi''(\psi(s,\delta),\delta)\Big)^2 + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big\}\Big|_{\delta=0}\\
=&2\Big(u_\varepsilon''(s,x)(|\varphi'(\psi(s,\delta),\delta)|^2)\Big|_{\delta=0} + u_\varepsilon'(s,x)(\varphi''(\psi(s,\delta),\delta)\Big|_{\delta=0}\Big)\Big(u_\varepsilon''(s,x)|\varphi'(\psi(s,\delta),\delta)|^2\Big|_{\delta=0}\\
&+ u_\varepsilon'(s,x)\frac{\partial}{\partial\delta}\varphi''(\psi(s,\delta),\delta)\Big|_{\delta=0}\Big)\\
=&2\Big(u_\varepsilon''(s,x)+0\Big)\Big(-2u_\varepsilon''(s,x)g'(s) - u_\varepsilon'(s,x)g''(s)\Big).
\end{aligned}$$

- Finally,

$$\begin{aligned}
&\frac{\partial}{\partial\delta}\Big(\psi'(s,\delta)e^{-\psi(s,\delta)}\Big\{\Big(u_\varepsilon''(s,x)|\varphi'(\psi(s,\delta),\delta)|^2 + u_\varepsilon'(s,x)\varphi''(\psi(s,\delta),\delta)\Big)^2 + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big\}\Big)\Big|_{\delta=0}\\
=&e^{-s}(g'(s)-g(s))\Big\{|u_\varepsilon''(s,x)|^2 + \varepsilon^2|\nabla u_\varepsilon(s,x)|^2 + \varepsilon^2|u_\varepsilon(s,x)|^p\Big\}\\
&-e^{-s}\Big(4|u_\varepsilon''(s,x)|^2g'(s) + 2u_\varepsilon'(s,x)u_\varepsilon''(s,x)g''(s)\Big).
\end{aligned}$$

Now by the minimality of $U_\varepsilon$ (see (45)), and the definitions of $L_\varepsilon$, $D_\varepsilon$, $I_\varepsilon$ and $I_\varepsilon'$ ((13), (16), (17) and (18)),

$$\int_0^\infty\Big(e^{-s}(g'(s)-g(s)\Big)L_\varepsilon(s)ds - \int_0^\infty\int_{\mathbb{R}^n} e^{-s}\Big(4D_\varepsilon g'(s) + 2I_\varepsilon'(s)g''(s)\Big)ds = 0.$$

But by integration by parts, (44), and since $g$ is bounded, $H_\varepsilon'(t) = -e^{-t}L_\varepsilon(t)$ (see (15)) and $H(t)\to 0$ when $t\to\infty$,

$$-\int_0^\infty e^{-s}g(s)L_\varepsilon(s)ds = g(s)H_\varepsilon(s)\Big|_0^\infty - \int_0^\infty H_\varepsilon(s)g'(s)ds = -\int_0^\infty H(s)g'(s)ds.$$

So we can rewrite our minimality condition as

$$\int_0^\infty\Big(e^{-s}L_\varepsilon(s) - H_\varepsilon(s)\Big)g'(s)ds - \int_0^\infty e^{-s}\Big(4D_\varepsilon(s)g'(s) + 2I_\varepsilon'(s)g''(s)\Big) = 0.$$

By definition of $g$ (43),

$$\int_0^\infty\Big(L_\varepsilon(s) - e^sH_\varepsilon(s) - 4D_\varepsilon(s)\Big)e^{-s}\eta(s)ds - 2\int_0^\infty I_\varepsilon'(s)e^{-s}\eta'(s) = 0.$$

We want to take $e^{-s}\eta$ as a single test function in $C_0^\infty(\mathbb{R}^+)$:

$$\int_0^\infty\Big(L_\varepsilon(s) - e^sH_\varepsilon(s) - 4D_\varepsilon(s) - 2I_\varepsilon'(s)\Big)(e^{-s}\eta(s))ds - 2\int_0^\infty I_\varepsilon'(s)(e^{-s}\eta(s))' = 0,$$

$\eta$ was arbitrary so $e^{-s}\eta$ is an arbitrary test function in $C_0^\infty(\mathbb{R}^+)$ and then

$$\frac{d}{dt}(-I_\varepsilon'(t)) = \frac{1}{2}L_\varepsilon(t) - \frac{1}{2}e^tH_\varepsilon(t) - 2D_\varepsilon(t) - I_\varepsilon'(t), \quad \text{in } \mathcal{D}'(\mathbb{R}^+).$$

By definition of $H'_\varepsilon$ (see (15)),

$$(e^t H_\varepsilon(t))' = e^t H_\varepsilon(t) - L_\varepsilon(t).$$

By definition of $E_\varepsilon$ (see (20)),

$$\begin{aligned} E'_\varepsilon(t) &= I'_\varepsilon(t) + \frac{d}{dt}(-I'_\varepsilon(t)) + \left(\frac{e^t}{2}H_\varepsilon(t)\right)' \\ &= I'_\varepsilon(t) + \frac{1}{2}L_\varepsilon(t) - \frac{1}{2}e^t H_\varepsilon(t) - 2D_\varepsilon(t) - I'_\varepsilon(t) + \frac{1}{2}e^t H_\varepsilon(t) - \frac{1}{2}L_\varepsilon(t) \\ &= -2D(t), \end{aligned}$$

and we get the result.

*Proof of Lemma 3.1.* In order to prove (21), note that

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|u''_\varepsilon|^2 dxdt \leq J_\varepsilon(u_\varepsilon).$$

In order to estimate $J_\varepsilon(u_\varepsilon)$, consider the competitor $h(t,x) = \alpha(x) + t\varepsilon\beta(x)$, that belongs to $\mathcal{U}^\varepsilon_{\alpha,\beta}$.Using the minimality of $u_\varepsilon$, we get

$$\begin{aligned} J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(h) &= \varepsilon^2 \int_0^\infty \int_{\mathbb{R}^n} e^{-t}\Big\{|\nabla h(t,x)|^2 + |h(t,x)|^p\Big\} dxdt \\ &\leq \varepsilon^2 \int_{\mathbb{R}^n}\Big(|\nabla\alpha(x)|^2 + |\alpha(x)|^p\Big)dx + C_1\varepsilon^3 \\ &\leq C\varepsilon^2, \end{aligned}$$

and we get (21). To prove (22) and (23), we apply Lemma 4.1 below with $h(t,x) = u_\varepsilon(t,x)$, obtaining

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|u'_\varepsilon(t,x)|^2 dxdt \leq 2\varepsilon^2 \int_{\mathbb{R}^n} |\beta(x)|^2 dx + C_2\varepsilon^2,$$

and (22) is established. (23) follows from Lemma 4.1 with $h = u$, combined with (22). Finally, (24) is a direct consequence of (21) and (22), combined with the Cauchy-Schwarz inequality. And the Lemma is proved.

In the previous proof, we used the following lemma

**Lemma 4.1.** *Let $h : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be a function such that, for every open set $A \subset \mathbb{R}^n$ of finite measure and for every $T > 0$, $h \in L^2((0,T)\times A)$ and $h' \in L^2((0,T)\times A)$. Then*

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|h(t,x)|^2 dxdt \leq 2\int_{\mathbb{R}^n} |h(0,x)|^2 dx + 4\int_0^\infty \int_{\mathbb{R}^n} e^{-t}|h'(t,x)|^2 dxdt.$$

*Proof of Lemma 4.1.* Given $T > 0$, for almost every $x \in \mathbb{R}^n$, the function $w(t) = h(t,x)$ belongs to $W^{1,2}(0,T)$. So we can integrate by parts and get

$$\begin{aligned} \int_0^T e^{-t}|w(t)|^2 dt &= -e^{-t}|w(t)|^2\Big|_0^T + 2\int_0^T e^{-t}w(t)w'(t)dt \\ &\leq |w(0)|^2 + 2\Big(\int_0^T e^{-t}|w(t)|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^T e^{-t}|w'(t)|^2 dt\Big)^{\frac{1}{2}}. \end{aligned}$$

Using $2\sqrt{ab} \leq a/2 + 2b$, we find

$$\int_0^T e^{-t}|w(t)|^2 dt \leq |w(0)|^2 + \frac{1}{2}\int_0^T e^{-t}|w(t)|^2 dt + 2\int_0^T e^{-t}|w'(t)|^2 dt,$$

which implies

$$\int_0^T e^{-t}|w(t)|^2 dt \leq 2|w(0)|^2 + 4\int_0^T e^{-t}|w'(t)|^2 dt.$$

This is true for almost every $x \in \mathbb{R}^n$, so integrating over $\mathbb{R}^n$ and letting $T \to \infty$, we get the inequality. And the lemma is proved.

## 5 Conclusion

Since the publication of [2], the authors extended this method to more general settings. Indeed in [3], Serra and Tilli proved that we can approach more general hyperbolic Cauchy problems by minimisation of elliptic energies. Concerning open questions on the topics of this report, probably the most challenging is to see whether it is possible to get the convergence without passing to subsequences.

# References


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